Exercises

1. Let K, L be fields. A function $f : K \to L$ is called homomorphism when $f(x + y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$, for any $x, y \in K$. Given a homomorphism $f : K \to L$ show that $f(0) = 0$. Also, show that only one of the following happens: $f(x) = 0, \forall x \in K$ or $f(1) = 1$ and f is injective.

2. Given a homomorphism $f : \mathbb{Q} \to \mathbb{Q}$. Show that only one of the following happens: $f(x) = 0, \forall x \in \mathbb{Q}$ or $f(x) = x, \forall x \in \mathbb{Q}$.

3. Explain why $\mathbb{Z}$, with its usual operations, is not a field.

4. Let K be an ordered field and $a, b \in K$. Show that $a^2 + b^2 = 0 \iff a = b = 0$.

5. Let $\mathcal{F}(X; K)$ denotes the set of all functions between X and K. Given $f, g \in \mathcal{F}(X; K)$, define the following operations on set the set $\mathcal{F}(X; K)$: $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$. Is $\mathcal{F}(X; K)$ a field?

6. Let x, y be positive elements of an ordered field K. Show that
$$x < y \iff x^{-1} > y^{-1}$$

7. Let $x \in K$ be a nonzero element in a ordered field K and $n \in \mathbb{N}$. Show that
$$(1 + x)^{2n} > 1 + 2n \cdot x$$

8. Let K be an ordered field and $a, x \in K$. If a and $a + x$ are positive, show that
$$(a + x)^n \geq a^n + n \cdot a^{n-1} \cdot x$$
.

9. Given an ordered field K, show the following are equivalent:
   a. K is Archimedean;
   b. $\mathbb{Z}$ is unbounded from below and from above;
   c. $\mathbb{Q}$ is unbounded from below and from above.

10. Given an ordered field K, show that K is Archimedean $\iff$ $\forall \epsilon > 0 \in K, \exists n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$.

11. Let $a > 1$ be an element of an Archimedean field K. Consider the function $f : \mathbb{Z} \to K$, given by $f(n) = a^n$. Show the following:
    a. $f(\mathbb{Z})$ is not bounded from above;
    b. $\inf f(\mathbb{Z}) = 0$.

12. Let $a, b, c, d \in \mathbb{Q}$. Show that
$$a + b\sqrt{2} = c + d\sqrt{2} \iff a = c \text{ and } b = d.$$

13. Let $a, b \in \mathbb{Q}$ be positive numbers. Show that

$$\sqrt{a} + \sqrt{b} \text{ is rational} \iff \text{both } \sqrt{a} \text{ and } \sqrt{b} \text{ are rational.}$$

14. Let $X = \{\,\frac{1}{n}\,;\, n \in \mathbb{N}\,\}$. Show that $\inf X = 0$.

15. Let $A \subseteq B \subseteq \mathbb{R}$ be nonempty bounded sets. Show that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

16. Let $A \subseteq \mathbb{R}$ be a bounded nonempty set. Show that

$$\sup -A = -\inf A.$$

17. Let $A \subseteq \mathbb{R}$ be a bounded nonempty set and $c > 0$, show that

$$\sup c \cdot A = c \cdot \sup A$$

.

18. Let $A, B \subseteq \mathbb{R}$ be bounded nonempty sets. Show that

$$\sup(A + B) = \sup A + \sup B;$$

and similarly, show that

$$\sup(A \cdot B) = \sup A \cdot \sup B,$$

where $A \cdot B = \{x \cdot y\,;\, x \in A, y \in B\}$.

19. Let $p > 1$ be a natural number. Show the set

$$X = \left\{ \frac{m}{p^n}\,;\, m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$.

20. A number $r \in \mathbb{R}$ is said to be **algebraic** if it is a root of a polynomial $p(x) \in \mathbb{Z}[x]$ with integral coefficients.

    a. Show that the set of all polynomials with integral coefficients, $\mathbb{Z}[x]$, is countable.
    b. Show that the set of all algebraic numbers is countable and dense in $\mathbb{R}$.

21. Let $X = \mathbb{R} - A$, where A is a countable subset of $\mathbb{R}$. Show that for each open interval (a, b), the intersection $X \cap (a, b)$ is uncountable. In particular, X is dense in $\mathbb{R}$.

22. A number $r \in \mathbb{R}$ is said to be **transcendental** if it's not algebraic. Show that the set of all transcendental numbers is uncountable and dense in $\mathbb{R}$.

23. Show that the set of algebraic numbers, usually denoted by $\overline{\mathbb{Q}}$, can be given a field structure. *This exercise assumes knowledge of Abstract algebra, you may skip it if you want.*

24. Give an example of open bounded nested intervals whose intersection is empty.

25. A **Dedekind cut** is an ordered pair (A, B), such that $A, B \subseteq \mathbb{Q}$ are nonempty, A doesn't have a maximum element, $\mathbb{Q} = A \cup B$, and $x < y$ for every $x \in A, y \in B$.

    a. Show that in a Dedekind cut (A, B) we have $\sup A = \inf B$.

    b. Let D be the set of all Dedeking cuts. Show that there is a bijection $f : D \to \mathbb{R}$.

26. Let X, Y be nonempty sets and $f : X \times Y \to \mathbb{R}$ a bounded function, i.e. $|f(x)| \leq c$. Let $f_1(x) = \sup\{f(x, y); y \in Y\}$ and $f_2(y) = \sup\{f(x, y); x \in X\}$. Show that

$$\sup_{x \in X} f_1(x) = \sup_{y \in Y} f_2(y).$$

In other words, sup commutes with itself:

$$\sup_x(\sup_y f(x, y)) = \sup_y(\sup_x f(x, y))$$

27. Generalize the exercise above and show that

$$\sup_y(\inf_x f(x, y)) \leq \inf_x(\sup_y f(x, y))$$

28. Let $x, y \in \mathbb{R}$ be positive numbers. Show that $\sqrt{x \cdot y} \leq \frac{x+y}{2}$

29. Show that the function $f : \mathbb{R} \to (-1, 1)$ defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$ is a bijection.

30. Let K be a complete ordered field. Let $1'$ denote the one of K. For each $n \in \mathbb{N}$, let $n' := \overbrace{1' + \ldots + 1'}^{n}$ and $(-n)' := -n'$. Define $f : \mathbb{R} \to K$ by $f(\frac{p}{q}) = \frac{p'}{q'}$ if $\frac{p}{q} \in \mathbb{Q}$, and $f(x) := \sup\left\{\frac{p'}{q'}; \frac{p'}{q'} < x\right\}$ if $x \in \mathbb{R} - \mathbb{Q}$. Show $f(x)$ is an isomorphism.

31. Let $f : \mathbb{R} \to \mathbb{R}$ be an automorphism. Show that $f(x) = x$, that is to say, f has to be the identity. Using Exercise 30, conclude that if K and L are complete ordered fields then there is a unique isomorphism between K and L.